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A theory for the rapid flow of identical, smooth, nearly elastic, spherical particles

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We focus attention on an idealized granular material comprised of identical, smooth, imperfectly elastic, spherical particles which is flowing at such a density and is being deformed at such a rate that particles interact only through binary collisions with their neighbours. Using general forms of the probability distribution functions for the velocity of a single particle and for the likelihood of binary collisions, we derive local expressions for the balance of mass, linear momentum and fluctuation kinetic energy, and integral expressions for the stress, energy flux and energy dissipation that appear in them. We next introduce simple, physically plausible, forms for the probability densities which contain as parameters the mean density, the mean velocity and the mean specific kinetic energy of the velocity fluctuations. This allows us to carry out the integrations for the stress, energy flux and energy dissipation and to express these in terms of the mean fields. Finally, we determine the behaviour of these fields as solutions to the balance laws. As an illustration of this we consider the shear flow maintained between two parallel horizontal plates in relative motion.

1. Introduction

Rapid deformations of dry, relatively dense granular materials occur in many industrial processes and geophysical phenomena. Such flows of interest here proceed at those densities and strain rates at which the impulsive forces in collisions between pairs of neighbouring particles are responsible for the transfer of momentum in the flowing material. At lower particle densities the transport of momentum by particle translations becomes important, while at lower rates and higher concentrations, multiple contacts of longer duration require the consideration of forces associated with the sliding of particles over their neighbours.

Roughly thirty years ago Bagnold (1954) considered the collisions between particles of a rapidly sheared granular material consisting of identical spherical grains. Bagnold argued that, because both the momentum exchanged in a collision and the frequency of collisions are proportional to the mean rate of shear, the shear stress and the normal stress must both be proportional to the square of the mean shear rate. The normal stress and shear stress measured in Bagnold's own experiments and the more recent experiments of Savage (1978) and Savage & Sayed (1980, 1982) on simple shear flows do depend on the mean rate of shear in this way. However, in more general shearing flows Bagnold's relations between the stresses and the mean shear rate require that the stresses vanish at points of the flow where the mean shear rate is zero. But at these points particle interactions may still persist as enduring contact forces between particles or as impulsive forces in collisions resulting from the fluctuations of the particle velocity about the mean. These fluctuations are an inevitable result of the collisions between particles being swept together by the mean flow. Bagnold did not consider these fluctuations except to assume that they were random and that the production of mean kinetic energy associated with them was balanced by dissipation into heat in collisions.

The importance of the velocity fluctuations and the means by which they could be included in a properly formulated continuum theory were first appreciated by Blinowski (1978) and Ogawa (1978). Blinowski exploited the analogy between the rapid deformation of a granular material and the turbulent flow of a compressible fluid, and, using modifications of arguments employed in the theories of turbulence, he derived local forms of the balance laws for the mean density and velocity, and for the second moment of the velocity fluctuations – the turbulent Reynolds stress. For a complete theory, the mean stress, and the flux and internal production of Reynolds stress must be related to the mean density, the mean velocity and the second moment of the velocity fluctuations. Ogawa adopted a somewhat simpler approach and introduced a fluctuation temperature in terms of the mean kinetic energy of the particles' velocity fluctuations. He proposed a balance law for this temperature that related its rate of change to its production by the mean flow, its flux from one point to another in the flow, and its dissipation into true thermal energy. Here again these last two quantities and the mean stress must be related by constitutive assumptions to the mean density, mean velocity and the fluctuation temperature in order to complete the theory. General forms of the constitutive relations for this theory or for Blinowski's formulation may be written down without difficulty. The problem is that they are so general that they are of little use. What would be most desirable are constitutive relations that, in addition to showing their dependence on the mean density and on the relevant means involving the velocity, exhibited explicitly the influence upon them of the known particle size, shape, mass, inelasticity, roughness etc. Such relations can only be obtained in advance by modelling the collisions between particles, determining the collisional changes in the physical quantities of interest, and averaging these over all possible collisions.

Ogawa (1978) has proposed such a scheme of modelling and averaging, and Ogawa, Umemura & Oshima (1980) have elaborated upon it. In their model a particle moves within a spherical surface that represents its neighbours. The radius of the sphere is determined by the particle density. Points on the sphere are assumed to be instantaneously moving with the mean velocity appropriate to their location. The particle moves relative to the mean flow with its fluctuation velocity. The distribution of this random variable is taken to be somewhat special; the magnitude of the velocity fluctuation is supposed to be constant and all orientations are assumed to be equally likely. In collisions, a fraction of the particles are assumed to adhere to the sphere, while the remainder rebound from it with a loss of energy and slip relative to its instantaneous motion. Ogawa *et al.* determine the total rate of change of fluctuation energy in such collisions by averaging over all possible collisions. This, with an estimate of the frequency of collisions, allows them to calculate the total rate of change of fluctuation energy and, finally, to infer how the mean stress and mean dissipation of fluctuation energy depend upon the collision parameters and upon the means of the density, fluctuation energy, and strain rate.

The flux of fluctuation energy is not given by this procedure, nor is it clear how the scheme might be extended in order to predict the form of this flux. This is the general disadvantage of the method. Too many simplifying assumptions are required to model the collisions and to carry out the averaging, and it is extremely difficult to see how these should be modified in order to extend or improve upon the predictions.

At this time it is clear that such extensions and improvements are necessary. Jenkins & Cowin (1979) have shown that the flux of fluctuation energy must be included in such a theory if it is to describe the steady, gravity-driven flow in a vertical channel. Without this flux there is no mechanism by which momentum can be transferred across the centreline of the channel, and the strain rate must vanish everywhere. Furthermore the theory of Ogawa *et al.* predicts stress levels in simple shear flow that are far too low (Lun, Savage & Jeffrey (1983)).

Recently Savage & Jeffrey (1981) have placed the problem of a rapidly deforming granular material in the context of the kinetic theory of dense gases. This immediately allows the use of numerous existing results regarding collisions and averaging, and avoids many of the difficulties confronted by Ogawa *et al.* One important difference between a kinetic theory for a classical dense gas and that for a rapidly deforming granular material is that in the granular material an inhomogeneity of the mean flow is necessary to force the collisions and to drive the velocity fluctuations. The temperature of a dense gas can also be influenced by the addition of heat throughout its interior or over its surface.† In Savage & Jeffrey's treatment, the importance of the mean deformation was reflected in the anisotropy of the distribution function that they proposed to govern the probability of collisions between pairs of particles. Collisions between particles being swept together by the mean flow were regarded as more likely than those between particles being swept apart. They considered a dense collection of identical spherical particles subjected to a rapid mean shear and, for perfectly elastic particles, calculated the components of the mean stress that result from the exchange of momentum in collisions.

The second important difference between the kinetic theories for a dense gas and a rapidly sheared granular material is that collisions between the particles of a granular material involve a loss of energy. Consequently, for smooth inelastic particles, Savage & Jeffrey's determination of the stress must be improved by incorporating the energy lost in collisions, and their calculations must be extended in order to deliver the rate at which fluctuation energy is dissipated into heat. Then, for example, in homogeneous, steady deformations, the fluctuation energy may be determined in terms of the mean rate of strain by equating the rate at which energy is supplied to the fluctuations by the mean flow to the rate at which fluctuation energy is dissipated into heat. This is the homogeneous, steady version of Ogawa's balance law for the rate of change of fluctuation energy.

Here we carry out a program outlined by Jenkins & Savage (1981) and develop a kinetic theory for rapid deformations of identical, smooth, nearly elastic, spherical particles. Using Maxwell's (1866) equations of transfer we derive the balance laws for mass, linear momentum and fluctuation energy. In these appear explicit expressions for the flux and production of linear momentum and fluctuation energy. These are given as integrals that involve the change of linear momentum and kinetic energy in a binary collision and the probability distribution function that governs the likelihood of such collisions. In order to evaluate these integrals we introduce a simplified form of the distribution function proposed by Savage & Jeffrey (1981) that

† There are, of course, situations involving granular materials in which the boundary can drive the fluctuations independently of the mean deformation or in which energy may be put into the fluctuations directly throughout the volume in the absence of a mean motion, but here we do not consider such active boundaries and ignore the inertia and viscosity of the interstitial gas.

is appropriate for nearly elastic particles. In it the anisotropy of the collisional pair distribution function is assumed to be linear in the mean rate of deformation. The form of the assumed anisotropy is identical with that predicted by numerical simulations of the molecular dynamics of rapidly sheared fluids (Ashurst & Hoover 1975; Evans & Watts 1980). Forms for the pressure tensor, the flux of fluctuation energy and the dissipation of fluctuation energy are calculated that exhibit the dependence of these on the mass, diameter and coefficient of restitution of the particles and on the means of the density, fluctuation energy and rate of deformation.

Using the derived balance laws and constitutive relations we consider a simple boundary-value problem in which the material is sheared between parallel horizontal plates. This is an idealization of the flow in the experiments of Savage & Sayed (1980, 1982). If the material is assumed to be sheared homogeneously, and realistic values for the coefficient of restitution and the parameter governing the anisotropy of the pair distribution are employed, the theory predicts the values of the shear and normal stress measured by Savage & Sayed. However, the assumption of homogeneity requires that there be no flux of fluctuation energy through the boundary – and it is not certain that this is the case.

2. Two-particle collisions

We consider the collision of two identical smooth spherical particles of mass m and diameter σ. Quantities associated with each sphere are distinguished by the subscript 1 or 2 and primes denote the values of these quantities following a collision.

The balance of linear momentum requires that the velocity vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ of the centre of each particle after a collision be related to those $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ before the collision by

$$m\boldsymbol{c}_1' = m\boldsymbol{c}_1 - \boldsymbol{J}, \quad m\boldsymbol{c}_2' = m\boldsymbol{c}_2 + \boldsymbol{J}, \tag{1}$$

where $\boldsymbol{J}$ is the impulse of the force exerted by particle 1 upon particle 2 in the collision.

The relative velocity $\boldsymbol{c}_{12}'$ of the centres of the particles after the collision is

$$\boldsymbol{c}_{12}' \equiv \boldsymbol{c}_1' - \boldsymbol{c}_2'. \tag{2}$$

Its component normal to the plane of contact is supposed here to be related to the corresponding component prior to the collision by

$$(\boldsymbol{k}\cdot\boldsymbol{c}_{12}') = -e(\boldsymbol{k}\cdot\boldsymbol{c}_{12}), \tag{3}$$

where $\boldsymbol{k}$ is the unit vector directed from the centre of the first particle to that of the second particle at contact, and e is the familiar coefficient of restitution. Depending upon the material of the particle, e may range from zero to one. When e equals one, the relative velocities of the centres of the particles is reversed upon collision and energy is conserved. Values of e less than one involve the dissipation of energy.

With (3) and (1) the value of the impulse $\boldsymbol{J}$ may be expressed in terms of $\boldsymbol{c}_{12}$ as

$$\boldsymbol{J} = \tfrac{1}{2}m(1+e)\,(\boldsymbol{k}\cdot\boldsymbol{c}_{12})\,\boldsymbol{k}. \tag{4}$$

Consequently the values of the particles' velocities are given in terms of their values before the collision by

$$\boldsymbol{c}_1' = \boldsymbol{c}_1 - \tfrac{1}{2}(1+e)\,(\boldsymbol{k}\cdot\boldsymbol{c}_{12})\,\boldsymbol{k}, \tag{5}$$

$$\boldsymbol{c}_2' = \boldsymbol{c}_2 + \tfrac{1}{2}(1+e)\,(\boldsymbol{k}\cdot\boldsymbol{c}_{12})\,\boldsymbol{k}. \tag{6}$$

The energy change ΔE in the collision is

$$\Delta E = \tfrac{1}{2}m(c_1'^2 + c_2'^2) - \tfrac{1}{2}m(c_1^2 + c_2^2), \tag{7}$$

where, for example, $c_1^2 \equiv \boldsymbol{c}_1 \cdot \boldsymbol{c}_1$. With (5) and (6), the change in energy may be written in terms of $\boldsymbol{c}_{12}$ as

$$\Delta E = \tfrac{1}{4}m(e^2-1)\,(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^2. \tag{8}$$

3. Statistical preliminaries

The statistics of the spatial arrangement of pairs of particles is governed by the two-particle configurational distribution function $n^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$, defined so that

$$n^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)\,\mathbf{d}\boldsymbol{r}_1\,\mathbf{d}\boldsymbol{r}_2 \tag{9}$$

is the probable number of pairs of particles with one in each of the volume elements $\mathbf{d}\boldsymbol{r}_1$ and $\mathbf{d}\boldsymbol{r}_2$ centred at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively. As defined, $n^{(2)}$ is unaffected by interchanging the positions of the two particles. When the mean flow is homogeneous, $n^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is isotropic and depends only upon $d \equiv |\boldsymbol{r}_2 - \boldsymbol{r}_1|$. In this event a radical distribution function $g_0(d)$ is defined by

$$g_0(d) \equiv \frac{n^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)}{n^2},$$

where n is the uniform number density of particles. An example of a particular radial distribution function that is important in what follows is that determined numerically by Carnahan & Starling (1969) for a fluid of identical hard spheres at contact, $d = \sigma$. It is given as a function of the solid-volume fraction $\nu = \frac{1}{6}\pi n \sigma^3$ by

$$g_0(\nu) = \frac{1}{1-\nu} + \frac{3\nu}{2(1-\nu)^2} + \frac{\nu^2}{2(1-\nu)^3}. \tag{10}$$

The statistics of the binary collisions are determined by the complete pair-distribution function $f^{(2)}$, a function of two particles' velocities, positions and the time defined so that

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2, t)\,\mathbf{d}\boldsymbol{c}_1\,\mathbf{d}\boldsymbol{r}_1\,\mathbf{d}\boldsymbol{c}_2\,\mathbf{d}\boldsymbol{r}_2 \tag{11}$$

is, at time t, the probable number of pairs of particles located in the volume elements $\mathbf{d}\boldsymbol{r}_1$, $\mathbf{d}\boldsymbol{r}_2$ centred at the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, and having velocities in the ranges $\mathbf{d}\boldsymbol{c}_1$ and $\mathbf{d}\boldsymbol{c}_2$ at $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. When this distribution is integrated over all velocities, by definition, $n^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is regained:

$$\int f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2, t)\,\mathbf{d}\boldsymbol{c}_1\,\mathbf{d}\boldsymbol{c}_2 = n^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2, t). \tag{12}$$

In a similar way a single-particle velocity distribution function $f^{(1)}(\boldsymbol{c}, \boldsymbol{r}, t)$ is defined so that $f^{(1)}(\boldsymbol{c}, \boldsymbol{r}, t)\,\mathbf{d}\boldsymbol{c}$ is the probable number of particles per unit volume at $\boldsymbol{r}$ and t with velocities in the element $\mathbf{d}\boldsymbol{c}$ at $\boldsymbol{c}$. Integrating $f^{(1)}$ over all velocities gives the local number density of particles

$$\int f^{(1)}(\boldsymbol{c}, \boldsymbol{r}, t)\,\mathbf{d}\boldsymbol{c} = n(\boldsymbol{r}, t). \tag{13}$$

Given any property $\psi(\boldsymbol{c})$, its mean, or ensemble average, $\langle\psi\rangle$ is determined in terms of $f^{(1)}$ by

$$\langle\psi\rangle = \frac{1}{n}\int \psi(\boldsymbol{c}) f^{(1)}(\boldsymbol{c}, \boldsymbol{r}, t)\,\mathbf{d}\boldsymbol{c}; \tag{14}$$

so, for example, the mean velocity $\boldsymbol{u}$ is $\langle\boldsymbol{c}\rangle$.

4. Maxwell transport and the balance laws

Let $\psi(\boldsymbol{c})$ be any property of a particle, and focus attention on a volume element $\mathbf{d}\boldsymbol{r}$ fixed in space. In a time $\mathrm{d}t$ the total mean amount of ψ, $\langle n\psi\rangle$, in $\mathbf{d}\boldsymbol{r}$ changes for three reasons: because the velocity $\boldsymbol{c}$ of each particle varies with time; because particles bearing ψ enter and leave $\mathbf{d}\boldsymbol{r}$; and because of collisions between particles in $\mathbf{d}\boldsymbol{r}$. Thus (Reif 1965, §14.4)

$$\frac{\partial}{\partial t}\langle n\psi\rangle = n\langle \mathrm{D}\psi\rangle - \boldsymbol{\nabla}\cdot\langle n\boldsymbol{c}\psi\rangle + \mathbb{C}(\psi), \tag{15}$$

where

$$\mathrm{D}\psi \equiv \frac{\mathrm{d}\boldsymbol{c}}{\mathrm{d}t}\cdot\frac{\partial\psi}{\partial\boldsymbol{c}} = m^{-1}\boldsymbol{F}\cdot\frac{\partial\psi}{\partial\boldsymbol{c}} \tag{16}$$

and $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, t)$ is the external force acting on a particle; the divergence is the mean efflux of ψ from $\mathbf{d}\boldsymbol{r}$, and $\mathbb{C}(\psi)$ is the mean collisional rate of change of ψ per unit volume. In order to calculate $\mathbb{C}(\psi)$ we must examine further the details of a collision.

Consider a particle with velocity $\boldsymbol{c}_2$ located at $\boldsymbol{r}_2$. For a second particle travelling with velocity $\boldsymbol{c}_1$ to collide with it in a time interval $\mathrm{d}t$ in such a way that, at collision, the line of centres $\boldsymbol{r}_2 - \boldsymbol{r}_1 = \sigma\boldsymbol{k}$ is within the solid angle $\mathbf{d}\boldsymbol{k}$ centred at $\boldsymbol{k}$, it is necessary that the centre of the second particle lie in a collision cylinder of volume $\sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\mathbf{d}\boldsymbol{k}\,\mathrm{d}t$. Consequently, the probable number of collisions per unit time such that $\boldsymbol{r}_2$ is in a volume element $\mathbf{d}\boldsymbol{r}_2$, the velocities $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ of the particles are in the ranges $\mathbf{d}\boldsymbol{c}_1$ and $\mathbf{d}\boldsymbol{c}_2$, and $\boldsymbol{k}$ is in the solid angle $\mathbf{d}\boldsymbol{k}$ is (Chapman & Cowling 1970, §16.2)

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2)\,\sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\mathbf{d}\boldsymbol{k}\,\mathbf{d}\boldsymbol{c}_1\,\mathbf{d}\boldsymbol{c}_2\,\mathbf{d}\boldsymbol{r}_2. \tag{17}$$

As a result of a collision, the property $\psi_2 = \psi(\boldsymbol{c}_2)$ changes to $\psi_2' = \psi(\boldsymbol{c}_2')$. This, with (17), gives the collisional rate of change of ψ per unit volume, $\mathbb{C}(\psi)$, as

$$\iiint_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0} (\psi_2' - \psi_2) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2)\,\sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\mathbf{d}\boldsymbol{k}\,\mathbf{d}\boldsymbol{c}_1\,\mathbf{d}\boldsymbol{c}_2, \tag{18}$$

where $\boldsymbol{c}_{12}\cdot\boldsymbol{k} > 0$ indicates that the integration is to be taken over all values for which a collision is impending. To obtain a more symmetric form we note that a collision identical with that just considered occurs between a particle with velocity $\boldsymbol{c}_1$ at $\boldsymbol{r}_2$ and a particle with velocity $\boldsymbol{c}_2$ at $\boldsymbol{r}_2 + \sigma\boldsymbol{k}$. For this collision the probability corresponding to (17) is

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_2, \boldsymbol{c}_2, \boldsymbol{r}_2 + \sigma\boldsymbol{k})\,\sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\mathbf{d}\boldsymbol{k}\,\mathbf{d}\boldsymbol{c}_2\,\mathbf{d}\boldsymbol{c}_1\,\mathbf{d}\boldsymbol{r}_2. \tag{19}$$

In this, however, the property of the particle at $\boldsymbol{r}_2$ is $\psi(\boldsymbol{c}_1)$. Consequently, an equivalent expression for $\mathbb{C}(\psi)$ is

$$\iiint_{\boldsymbol{c}_{12}\cdot\boldsymbol{k}>0} (\psi_1' - \psi_1) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_2, \boldsymbol{c}_2, \boldsymbol{r}_2 + \sigma\boldsymbol{k})\,\sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\,\mathbf{d}\boldsymbol{k}\,\mathbf{d}\boldsymbol{c}_1\,\mathbf{d}\boldsymbol{c}_2. \tag{20}$$

The probability distribution in (20) may be related to that in (18) by shifting the pair of spatial points at which it is evaluated through the expansion

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_2, \boldsymbol{c}_2, \boldsymbol{r}_2 + \sigma\boldsymbol{k}) = f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2) + \sigma\boldsymbol{k}\cdot\boldsymbol{\nabla} f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2). \tag{21}$$

Using this in (20), adding the result to (18), and taking half of the sum gives $\mathbb{C}(\psi)$ in the form

$$\mathbb{C}(\psi) = -\boldsymbol{\nabla}\cdot\boldsymbol{\theta}(\psi) + \chi(\psi), \tag{22}$$

where

$$\boldsymbol{\theta}(\psi) = -\tfrac{1}{2}\sigma \iiint_{c_{12}\cdot k>0} \boldsymbol{k}(\psi_1' - \psi_1) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2)\, \sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{c}_1\, \mathrm{d}\boldsymbol{c}_2, \tag{23}$$

$$\chi(\psi) = \frac{1}{2} \iiint_{c_{12}\cdot k>0} (\psi_1' + \psi_2' - \psi_1 - \psi_2) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2)\, \sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{c}_1\, \mathrm{d}\boldsymbol{c}_2. \tag{24}$$

With (22), the general form of (15) is

$$\frac{\partial}{\partial t}\langle n\psi\rangle = n\langle \mathrm{D}\psi\rangle - \boldsymbol{\nabla}\cdot\langle n\boldsymbol{c}\psi\rangle - \boldsymbol{\nabla}\cdot\boldsymbol{\theta}(\psi) + \chi(\psi), \tag{25}$$

in which $\mathrm{D}\psi$ is given by (16), $\boldsymbol{\theta}(\psi)$ by (23), and $\chi(\psi)$ by (24).

Taking ψ to be m in (25) yields the local form of the conservation of mass:

$$\dot{\rho} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{26}$$

where $\rho = mn$ is the mass density and the dot indicates a time derivative calculated with respect to the mean motion.

Taking ψ to be $m\boldsymbol{c}$ in (25) and using (26) gives the local form of the balance of linear momentum

$$\rho\dot{\boldsymbol{u}} = -\boldsymbol{\nabla}\cdot\langle\rho\boldsymbol{C}\otimes\boldsymbol{C}\rangle - \boldsymbol{\nabla}\cdot\boldsymbol{P} + n\boldsymbol{F}, \tag{27}$$

where $\boldsymbol{C} \equiv \boldsymbol{c} - \boldsymbol{u}$, $\otimes$ denotes a tensor product, and $\boldsymbol{P}$ is the collisional pressure tensor given by

$$\boldsymbol{P} = -\tfrac{1}{2}m\sigma \iiint_{c_{12}\cdot k>0} (\boldsymbol{c}_1' - \boldsymbol{c}_1)\otimes\boldsymbol{k}\, f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2)\, \sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{c}_1\, \mathrm{d}\boldsymbol{c}_2. \tag{28}$$

Finally, taking ψ to be $\frac{1}{2}mc^2$ in (25), noting that

$$\theta(\tfrac{1}{2}mc^2) = \boldsymbol{u}\cdot\boldsymbol{P} + \boldsymbol{q}, \tag{29}$$

where

$$\boldsymbol{q} = -\tfrac{1}{4}m\sigma \iiint_{c_{12}\cdot k>0} \boldsymbol{k}(C_1'^2 - C_1^2) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2)\, \sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{c}_1\, \mathrm{d}\boldsymbol{c}_2, \tag{30}$$

and employing (26) and (27), yields the local form of the balance of fluctuation energy:

$$\tfrac{3}{2}\rho\dot{T} = -\boldsymbol{\nabla}\cdot\langle\tfrac{1}{2}\rho C^2\boldsymbol{C}\rangle - \mathrm{tr}\,(\boldsymbol{\nabla u}\langle\rho\boldsymbol{C}\otimes\boldsymbol{C}\rangle) - \boldsymbol{\nabla}\cdot\boldsymbol{q} - \mathrm{tr}\,(\boldsymbol{P}\boldsymbol{\nabla u}) - \gamma. \tag{31}$$

Here the fluctuation energy T is given in terms of the specific kinetic energy of the fluctuations by $3T = \langle C^2\rangle$, tr denotes the trace, and γ is the collisional rate of dissipation per unit volume,

$$\gamma = -\tfrac{1}{4}m \iiint_{c_{12}\cdot k>0} (c_1'^2 + c_2'^2 - c_1^2 - c_2^2) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2)\, \sigma^2(\boldsymbol{c}_{12}\cdot\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{c}_1\, \mathrm{d}\boldsymbol{c}_2, \tag{32}$$

where the energy change in a collision is given in terms of the coefficient of restitution e by (8).

Because of the presence of dissipation the standard techniques of the kinetic theory that exploit symmetries present in conservative collisions could not be used to obtain the balance laws and the integral forms of the constitutive relations. In particular, the relatively simple argument leading to the integral forms for the collisional flux (23) and the collisional production (24) is crucial to the development of the present theory.

Contributions to the flux of linear momentum and fluctuation energy arise from two sources, the transport of these quantities by particles moving between collisions, and their transfer in collisions. For the dense collections of particles that we consider here, collisional transfer dominates and transport is negligible. Consequently, in what follows we make an insignificant error in ignoring the first term on the right-hand side of (27) and the first two terms on the right-hand side of (31).

5. The collisional pair-distribution function

In order to calculate explicit expressions for $\boldsymbol{P}$, $\boldsymbol{q}$, and γ it is necessary to determine the form of the complete pair-distribution function $f^{(2)}$ at collision. In principle this function should be obtained as the solution of an evolution equation, perhaps similar in form to the Boltzmann equation governing the evolution of $f^{(1)}$ in the kinetic theory of gases. However, even if we were secure in proposing the form of the evolution equation for $f^{(2)}$ in these dense dissipative systems, the difficulties of obtaining and interpreting solutions would be at least as great as for the Boltzmann equation.

Here we avoid these difficulties and introduce a simple, physically plausible form for $f^{(2)}$ at collision. The underlying idea is that, when an inhomogeneous mean flow is present in these dense arrays of particles, collisions between those neighbouring particles being swept towards each other by spatial variations in the mean flow are more likely than for particles being swept away from each other. The proposed form of $f^{(2)}$ exhibits this collisional anisotropy in, perhaps, the simplest possible way; likewise, its dependence upon the particle velocities is given in terms of elementary but plausible velocity distributions for each particle. The resulting form of $f^{(2)}$ contains the mean fields n, $\boldsymbol{u}$, and T as parameters. We require that these five functions be solutions of the Maxwell transport equations (26), (27) and (31) corresponding to the conservation of mass, the balance of linear momentum and the balance of fluctuation kinetic energy.

Following Savage & Jeffrey (1981), we first adopt a slight generalization of the assumption of molecular chaos and suppose that the distribution function can be expressed as the product of a normalized pair-distribution function $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ and the single particle velocity distribution function for each particle,

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2) = g(\boldsymbol{r}_1, \boldsymbol{r}_2) f^{(1)}(\boldsymbol{c}_1, \boldsymbol{r}_1) f^{(1)}(\boldsymbol{c}_2, \boldsymbol{r}_2), \tag{33}$$

where $\boldsymbol{r}_2 - \boldsymbol{r}_1 = \sigma \boldsymbol{k}$,

$$g(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{n^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2)}{n_1 n_2}, \tag{34}$$

and the subscripts 1 and 2 on mean fields indicate that these are to be evaluated at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively.

We next suppose with Savage & Jeffrey that the single-particle velocity distributions are Maxwellian about the mean velocity. Then, for example,†

$$f^{(1)}(\boldsymbol{c}_1, \boldsymbol{r}_1) = n_1 \left(\frac{1}{2\pi T_1}\right)^{\frac{3}{2}} \exp\left[-\frac{(c_1 - u_1)^2}{2T_1}\right]. \tag{35}$$

Finally, as do Savage & Jeffrey, we argue that, in order for $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ to exhibit an anisotropy due to the presence of the mean flow, it should depend not only upon ν, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ but also upon T, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Here ν and T are evaluated at the point of contact. For form invariance under rigid motions the dependence of g on its vector arguments must be restricted to the inner products of $\boldsymbol{r}_2 - \boldsymbol{r}_1 = -\sigma \boldsymbol{k}$ and $\boldsymbol{u}_2 - \boldsymbol{u}_1 \equiv \boldsymbol{u}_{21}$. Then for dimensional homogeneity the arguments of g can only be ν, $\boldsymbol{k} \cdot \boldsymbol{u}_{21}/T^{\frac{1}{2}}$, and u_{21}^2/T. In what follows we restrict attention to situations in which the magnitude of $\boldsymbol{u}_{21}$ is small relative to $T^{\frac{1}{2}}$. In this limit we adopt for $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ its most general linear anisotropic form

$$g(\boldsymbol{r}_1, \boldsymbol{r}_2) = g_0 \left[1 - \frac{\alpha \boldsymbol{k} \cdot \boldsymbol{u}_{21}}{(\pi T)^{\frac{1}{2}}}\right], \tag{36}$$

† The distribution function given by Savage & Jeffrey must be corrected by replacing $\overline{v^2}$ by $\frac{2}{3}\overline{v^2}$ throughout (Savage & Jeffrey 1982).

where the dependence of g_0 upon ν is assumed to be given by (10), and α is an arbitrary function of ν that remains to be specified. For example, a linearization of a form for $g(\boldsymbol{r}_1, \boldsymbol{r}_2)$ proposed by Savage & Jeffrey gives a constant $\alpha = 1$. For a homogeneous shear flow the form of the pair-distribution function (36) reduces to that obtained by Ashurst & Hoover (1975) and Evans & Watts (1980) in numerical simulations of the dynamics of particles subjected to a mean shear. A later analysis of such a flow here indicates that for nearly elastic particles the anisotropic term is sufficiently small to justify the linear expansion.

With these assumptions the complete pair-distribution function at collision has the form

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2) = g_0\left(\frac{1}{2\pi}\right)^3 \frac{n_1 n_2}{(T_1 T_2)^{\frac{3}{2}}}\left[1 - \frac{\alpha \boldsymbol{k}\cdot\boldsymbol{u}_{21}}{(\pi T)^{\frac{1}{2}}}\right]\exp\left\{-\left[\frac{(c_1-u_1)^2}{2T_1} + \frac{(c_2-u_2)^2}{2T_2}\right]\right\}. \quad (37)$$

6. Constitutive relations

In order to evaluate the collision integrals (28), (30), and (32) we adopt the form (37) for $f^{(2)}$ and expand it in terms of a Taylor series about the point of contact $\boldsymbol{r}$. This allows the evaluation of the mean fields and their derivatives at this point. Upon discarding terms involving spatial derivatives higher than the first, this expansion yields the approximation (Chapman & Cowling 1970, §16.31)

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r} - \tfrac{1}{2}\sigma\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r} + \tfrac{1}{2}\sigma\boldsymbol{k}) = g_0 f_{10}^{(1)} f_{20}^{(1)}\left\{1 - \frac{\alpha\sigma}{(\pi T)^{\frac{1}{2}}}[(\boldsymbol{k}\cdot\boldsymbol{\nabla})\boldsymbol{u}]\cdot\boldsymbol{k} + \frac{\sigma}{2}(\boldsymbol{k}\cdot\boldsymbol{\nabla})\left[\ln\frac{f_{20}^{(1)}}{f_{10}^{(1)}}\right]\right\}, \quad (38)$$

where, for example,
$$f_{10}^{(1)} \equiv n\left(\frac{1}{2\pi T}\right)^{\frac{3}{2}}\exp\left[-\frac{(c_1-u)^2}{2T}\right]. \quad (39)$$

Employing (38) in the collision integrals and carrying out the integrations in a manner parallel to that of Chapman & Cowling (1970, §§16.41 and 16.42, with the aid of the integrals evaluated in their §§1.42 and 16.8 and others given in the appendix) we obtain the constitutive relations

$$\gamma = \frac{\kappa(1-e)}{2\sigma^2}\left[12T - (3\pi + 4\alpha)\sigma\left(\frac{T}{\pi}\right)^{\frac{1}{2}}(\mathrm{tr}\,\boldsymbol{D})\right], \quad (40)$$

where
$$\kappa = 2\rho\nu g_0\sigma(1+e)\left(\frac{T}{\pi}\right)^{\frac{1}{2}}, \quad 2\boldsymbol{D} = \boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\mathrm{T}},$$

$$\boldsymbol{q} = -\kappa\boldsymbol{\nabla}T, \quad (41)$$

$$\boldsymbol{P} = \pi^{\frac{1}{2}}\kappa\sigma^{-1}T^{\frac{1}{2}}\boldsymbol{I} - \tfrac{1}{5}\kappa(2+\alpha)[(\mathrm{tr}\,\boldsymbol{D})\boldsymbol{I} + 2\boldsymbol{D}] \quad (42)$$

(here $\boldsymbol{I}$ is the unit tensor).

The forms of the rate of dissipation (40) and the pressure tensor (42) differ from those calculated by Ogawa *et al.* (1980). This is due to the differences in modelling the collisions and carrying out the averages. The flux of fluctuation energy is essentially identical with that of the heat flux calculated by Chapman & Cowling (1970), and the pressure tensors of the two theories are similar. However, this resemblance of results is not due to any similarity in method but to a superficial similarity in the structure of the distribution function (38) and that derived by Chapman & Cowling as an approximate solution to the Boltzmann equation.

7. Shearing between parallel horizontal plates

To illustrate the nature of the boundary-value problems that result from the balance laws and constitutive theory, we consider shear flow maintained between parallel flat plates a fixed distance L apart. The plates are supposed to be horizontal and in relative motion.

Of course, a complete formulation of the boundary-value problem requires the conditions on, say, the mean velocity and the energy flux be specified on the surface of the plates. However, as yet, we are not certain of the exact form of these boundary conditions. Videotapes of experiments carried out by Savage & Sayed show that even at a roughened boundary the mean velocity of the particles is not that of the boundary. Some slip seems always to occur. The videotapes also show that the velocity fluctuations of the particles in the neighbourhood of the boundary are suppressed. This indicates a flux of energy through the boundary. In general, such a flux should depend upon the fluctuation energy near the surface, the material of the surface, and its roughness.

A derivation of the correct boundary conditions on the velocity and energy flux appears to require a detailed analysis of the collisional transfer between the boundary and neighbouring particles similar to and perhaps more complicated than that just carried out. In the absence of such an analysis and for the purpose of illustration, we shall suppose that the mean velocity and the energy flux are specified at a boundary but defer consideration of how these specifications are related to the velocity of the boundary, the velocity fluctuations near it, the material of the boundary, and its roughness.

An appropriate orientation of the x- and z-axes of a rectangular Cartesian system insures that the density, the non-vanishing x-component u of the mean velocity, and the specific fluctuation energy are functions of z alone. In this event, the conservation of mass (26) is identically satisfied; the balance of linear momentum (27) reduces to

$$0 = -P'_{xz}, \tag{43}$$

$$0 = -P'_{zz} - \rho G, \tag{44}$$

where the prime indicates a derivative with respect to z, and G is the gravitational acceleration; and the balance of fluctuation energy (31) becomes

$$0 = -q'_z - P_{xz} u' - \gamma. \tag{45}$$

From (42), the relevant components of the pressure tensor are

$$P_{xz} = -\tfrac{1}{5}\kappa(2+\alpha)\, u', \tag{46}$$

$$P_{zz} = P_{zz} = \pi^{\frac{1}{2}}\kappa\sigma^{-1}T^{\frac{1}{2}}; \tag{47}$$

from (41) the energy flux is

$$q_z = -\kappa T'; \tag{48}$$

and from (40) the dissipation is given by

$$\gamma = 6\sigma^{-2}(1-e)\,\kappa T. \tag{49}$$

Because of the presence of gravity, there is no symmetry about the centreline of the flow; so we choose the origin of the z-axis at the bottom plate. Without loss of generality we may suppose that the mean velocity of the particles is zero at $z = 0$ and U at $z = L$. In general the energy flux through the top and bottom boundaries will be different because the velocity fluctuations at the upper and lower plate will differ. Because of this, in the more complicated boundary-value problem involving gravity we only obtain the ordinary differential equations governing the mean density, velocity and fluctuation energy and indicate their solutions. When gravity can be ignored, we exhibit solutions to the differential equations that depend upon U and the flux Q of fluctuation energy at the upper and lower boundaries.

The horizontal component (43) of the linear momentum balance may be integrated immediately and the result used with the constitutive relation (46) to obtain

$$\tfrac{1}{5}\kappa(2+\alpha)\,u' = S, \tag{50}$$

where S is a constant equal to the shear stress applied to the flow at the top plate.

An approximate integral of the vertical component (44) of the momentum balance (that might serve as the first step in an iterative scheme) may be obtained by replacing the local value of the density ρ by its constant mean value $\bar{\rho}$ over the width of the flow. This, used with the constitutive relation (47), yields

$$\pi^{\frac{1}{2}}\kappa\sigma^{-1}T^{\frac{1}{2}} = -\bar{\rho}Gz+N, \tag{51}$$

where N is the pressure at the bottom plate.

Employing the shear stress S and the expressions (48) and (49) for the energy flux and the dissipation, the energy balance (45) may be written as

$$(\kappa T')'+Su'-6\sigma^{-2}(1-e)\,\kappa T = 0. \tag{52}$$

From (51), κ may be expressed in terms of z and T and this used in (50) to obtain a similar representation for u'. With these, (52) may be rewritten as

$$\left[\frac{N-\bar{\rho}Gz}{T^{\frac{1}{2}}}\,T'\right]'+\frac{5\pi S^2 T^{\frac{1}{2}}}{\sigma^2(2+\alpha)\,(N-\bar{\rho}Gz)}-\frac{6(1-e)}{\sigma^2}\,(N-\bar{\rho}Gz)\,T^{\frac{1}{2}} = 0. \tag{53}$$

Finally, upon introducing the new independent variable $w = T^{\frac{1}{2}}$ and the dimensionless vertical coordinate

$$s = \sigma^{-1}[3(1-e)]^{\frac{1}{2}}\left(\frac{N}{\bar{\rho}G}-z\right), \tag{54}$$

(53) may be written as

$$\ddot{w}+\frac{1}{s}\,\dot{w}-\left(1-\frac{\mu^2}{s^2}\right)w = 0, \tag{55}$$

where a dot denotes a derivative with respect to s and

$$\mu^2 = \frac{5\pi}{2(2+\alpha)}\left(\frac{S}{\bar{\rho}G\sigma}\right)^2. \tag{56}$$

Although it is remarkable that (55) governing the specific fluctuation energy is linear, its solutions are Bessel functions of imaginary order and imaginary argument (Watson 1966, p. 47). Since these are, apparently, not tabulated, the linearity of (55) is of little utility. In principle the solution for T allows the integration of (50) for u, the integration of (51) for ρ, and the imposition of the boundary conditions. Physically, such solutions are important in interpreting experiments in which the vertical forces applied to the boundaries of a shear cell are of the same order as the weight of material contained in the cell. Experiments that include data in this range, corresponding to relatively low shear rates, have been reported by Savage & Sayed (1980, 1982). Practically, the determination of the detailed form of such solutions must await a better understanding of the boundary conditions.

In situations in which the shear rates are so high that the vertical forces that must be applied to a shear cell are much greater than the weight of the material contained in the cell, a far simpler analysis of the flow between parallel plates is possible. The largest number of experiments carried out by Savage & Sayed (1980, 1982) fall in this range. The analysis exploits the symmetry present when gravity may be ignored.

In this case we choose the origin of the z-axis at the centre of the flow and adopt the boundary conditions

$$u(\tfrac{1}{2}L) = -u(-\tfrac{1}{2}L) = U, \tag{57}$$

$$q_z(\tfrac{1}{2}L) = -q_z(-\tfrac{1}{2}L) = Q. \tag{58}$$

When gravity is negligible (50) is unchanged, but (51) becomes

$$\pi^{\frac{1}{2}}\kappa\sigma^{-1}T^{\frac{1}{2}} = N, \tag{59}$$

where N is now the constant pressure throughout the flow. Then (59) and (50) may be used to eliminate κ and u' from the energy balance (52), and the resulting equation may be written in the form

$$\ddot{w}+\lambda w = 0, \tag{60}$$

where a dot here denotes a derivative with respect to the non-dimensional vertical coordinate $s = z/L$ and

$$\lambda = \left(\frac{L}{\sigma}\right)^2\left[\frac{5\pi}{2(2+\alpha)}\left(\frac{S}{N}\right)^2 - 3(1-e)\right]. \tag{61}$$

If λ is zero, the specific fluctuation energy is uniform and there is no flux of energy through the boundaries. Then (59) used with the definition of κ relates the uniform density to the specific fluctuation energy, and (59) and (50) used together with the boundary conditions (57) determine the relationship between T and the constant shear rate $2U/L$:

$$T^{\frac{1}{2}} = 2\left(\frac{2+\alpha}{5\pi^{\frac{1}{2}}}\right)\frac{N}{S}\frac{\sigma}{L}U. \tag{62}$$

The ratio of S to N is determined by (61) upon setting λ equal to zero, in which event (62) may be written as

$$T^{\frac{1}{2}} = 2\left[\frac{2+\alpha}{30(1-e)}\right]^{\frac{1}{2}}\frac{\sigma U}{L}. \tag{63}$$

Because κ is proportional to $T^{\frac{1}{2}}$, in this homogeneous shearing flow both the shear stress S, from (50), and the normal stress N, from (59), are proportional to the square of the shear rate. This need not be the case in more general flows. Also the ratio $\sigma U/LT^{\frac{1}{2}}$, which could not be determined by Savage & Jeffrey (1981), is given through (63) in terms of the coefficient of restitution e and α. This ratio is small when the particles are nearly elastic and e is close to one. In this case the anisotropic term in the expansion (36) of the pair-distribution function is also small. This is the reason that we have restricted our attention to nearly elastic particles.

The experiments of Savage & Sayed indicate that at the relatively high rates of shear the ratio of the shear stress to the normal stress is approximately constant. This is consistent with the behaviour predicted by (61) for the homogeneous flow if α is nearly constant. In fact, over the range of strain rates where the ratio is constant, the measured values of the shear stress and normal stress can be reproduced by choosing $\alpha = 1$ and $e = 0.9$ (Lun, *et al.* 1983). However, we anticipate that only in exceptional cases will there be no flux of energy through a boundary.

In order to treat the general case we suppose that λ is positive and adopt the simplification of constant α. Then the elementary solution of the boundary-value problem (58) and (60) is

$$T^{\frac{1}{2}} = 2\left(\frac{\pi}{\lambda}\right)^{\frac{1}{2}}\frac{L}{\sigma}\frac{Q}{N}\frac{\cos(\lambda^{\frac{1}{2}}z/L)}{\sin(\frac{1}{2}\lambda^{\frac{1}{2}})}. \tag{64}$$

With this, the analogue of (62) may be integrated to

$$u = \frac{10\pi}{\lambda(2+\alpha)}\frac{S}{N}\left(\frac{L}{\sigma}\right)^2 \frac{Q}{N}\frac{\sin(\lambda^{\frac{1}{2}}z/L)}{\sin(\frac{1}{2}\lambda^{\frac{1}{2}})}, \tag{65}$$

where the integration constant has been set equal to zero to ensure that u is odd in z. The boundary condition (57) applied to the solution (65) forces the conclusion that, if U and S have the same sign, then Q must be positive, and, with the definition (61) of λ, leads to the quadratic equation

$$\left(\frac{S}{N}\right)^2 - \frac{4Q}{UN}\frac{S}{N} - \frac{6(1-e)(2+\alpha)}{5\pi} = 0 \tag{66}$$

for the determination of S/N.

In the event that λ is negative, the alternative to (64) is easily obtained in terms of cosh $(|\lambda|^{\frac{1}{2}} z/L)$. Then, if U and S have the same sign, Q is negative.

8. Concluding remarks

We have presented what may be the simplest possible kinetic theory for the rapid deformations of a granular material. This simple theory gives qualitative and quantitative predictions in agreement with the experiments known to us on rapid deformations of identical, smooth, nearly elastic, spherical particles at moderate concentrations. However the greatest benefit in developing such a kinetic theory is in clarifying the questions of what assumptions must be abandoned or what ingredients must be added in order to improve the theory and to extend it to more complicated physical situations.

At this stage it is not difficult to see how to extend the theory to less elastic spheres, to rough, spinning disks or spheres, or to mixtures of spheres with different diameters. These calculations are all in progress.

A more difficult problem is the derivation of appropriate boundary conditions based upon a description of the geometry, physical properties, and motion of a bounding surface; the details of a collision between a particle and the boundary; and a statistical characterization of the likelihood of such collisions. Here it may be necessary to introduce the second moments of the velocity fluctuations into the theory. There also are difficulties in extending the present theory to include the inertial and viscous properties of the gas or fluid surrounding the particles. It seems clear that in the present context pneumatic effects could serve as a source of fluctuation energy while viscosity would contribute to the rate of dissipation of the fluctuations. Ackermann & Shen (1982) and Shen & Ackermann (1982) have been relatively successful in incorporating viscous dissipation in a scheme of modelling and averaging similar to that employed by Ogawa and his collaborators, but it is not yet clear how to include the inertial and viscous properties of the fluid phase in a kinetic theory for granular materials.

Finally there are the questions of the form of the evolution equation for the distribution $f^{(2)}$ and of the method by which multiple and sliding contacts can be treated. Questions similar to these have been the subject of research in the classical kinetic theory since its beginning, but it remains to be seen how the results of this research can be applied in a kinetic theory for granular materials.

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Appendix

Here we provide some integrations with respect to $\boldsymbol{k}$ to supplement those given by Chapman & Cowling (1970, §16.8). These are used in deriving the constitutive relations (40), (41) and (42) from the integrals (32), (30) and (28) respectively, when the pair-distribution function is given by (38).

Following Chapman and Cowling we introduce an orthonormal triad of base vectors $\boldsymbol{h}$, $\boldsymbol{i}$, and $\boldsymbol{j}$ and choose $\boldsymbol{h}$ parallel to $\boldsymbol{c}_{12}$. We suppose that θ and ϕ are the polar angles of $\boldsymbol{k}$ with respect to $\boldsymbol{h}$ and the plane of $\boldsymbol{h}$ and $\boldsymbol{i}$ respectively, so that

$$\boldsymbol{k} = \boldsymbol{h}\cos\theta + \boldsymbol{i}\sin\theta\cos\phi + \boldsymbol{j}\sin\theta\sin\phi. \tag{A 1}$$

Then $\boldsymbol{c}_{12}\cdot\boldsymbol{k} = \cos\theta$, and the element of solid angle $\mathbf{d}\boldsymbol{k}$ is given by

$$\mathbf{d}\boldsymbol{k} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{A 2}$$

The integrals are to be taken over all values of $\boldsymbol{k}$ for which $\boldsymbol{c}_{12}\cdot\boldsymbol{k}$ is positive, so in them θ varies between 0 and $\frac{1}{2}\pi$, while ϕ ranges from 0 to 2π. Consequently all terms in the integrand containing odd powers of $\sin\phi$ or $\cos\phi$ contribute nothing, and will, for brevity, be omitted.

So, for example,

$$\int \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^3\,\mathbf{d}\boldsymbol{k} = c_{12}^3\iint \boldsymbol{h}\cos^4\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{2\pi}{5}c_{12}^2\boldsymbol{c}_{12}; \tag{A 3}$$

while, if $\boldsymbol{v}$ is any vector independent of θ and ϕ,

$$\int(\boldsymbol{v}\cdot\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^3\,\mathbf{d}\boldsymbol{k} = c_{12}^3\iint(\boldsymbol{v}\cdot\boldsymbol{h})\cos^4\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \tfrac{2}{5}\pi c_{12}^2(\boldsymbol{v}\cdot\boldsymbol{c}_{12}), \tag{A 4}$$

$$\begin{aligned}
&\int(\boldsymbol{k}\cdot\boldsymbol{v})^2(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^3\,\mathbf{d}\boldsymbol{k}\\
&= c_{12}^3\iint[(\boldsymbol{v}\cdot\boldsymbol{h})^2\cos^2\theta + (\boldsymbol{v}\cdot\boldsymbol{i})^2\sin^2\theta\cos^2\phi + (\boldsymbol{v}\cdot\boldsymbol{j})^2\sin^2\theta\sin^2\phi]\cos^3\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \tfrac{1}{12}\pi c_{12}^3[4(\boldsymbol{v}\cdot\boldsymbol{h})^2 + (\boldsymbol{v}\cdot\boldsymbol{i}) + (\boldsymbol{v}\cdot\boldsymbol{j})^2].
\end{aligned} \tag{A 5}$$

Finally,

$$\begin{aligned}
&\int \boldsymbol{k}\otimes\boldsymbol{k}(\boldsymbol{v}\cdot\boldsymbol{k})^2(\boldsymbol{k}\cdot\boldsymbol{c}_{12})\,\mathbf{d}\boldsymbol{k}\\
&= c_{12}\iint\{[\boldsymbol{h}\otimes\boldsymbol{h}\cos^2\theta + \boldsymbol{i}\otimes\boldsymbol{i}\sin^2\theta\cos^2\phi + \boldsymbol{j}\otimes\boldsymbol{j}\sin^2\theta\sin^2\phi]\\
&\quad\cdot[(\boldsymbol{v}\cdot\boldsymbol{h})^2\cos^2\theta + (\boldsymbol{v}\cdot\boldsymbol{i})^2\sin^2\theta\cos^2\phi + (\boldsymbol{v}\cdot\boldsymbol{j})^2\sin^2\theta\sin^2\phi]\\
&\quad + 2(\boldsymbol{v}\cdot\boldsymbol{h})[(\boldsymbol{h}\otimes\boldsymbol{i} + \boldsymbol{i}\otimes\boldsymbol{h})(\boldsymbol{v}\cdot\boldsymbol{i})\cos^2\phi\\
&\quad + (\boldsymbol{h}\otimes\boldsymbol{j} + \boldsymbol{j}\otimes\boldsymbol{h})(\boldsymbol{v}\cdot\boldsymbol{j})\sin^2\phi]\sin^2\theta\cos^2\theta\\
&\quad + 2(\boldsymbol{i}\otimes\boldsymbol{j} + \boldsymbol{j}\otimes\boldsymbol{i})(\boldsymbol{v}\cdot\boldsymbol{i})(\boldsymbol{v}\cdot\boldsymbol{j})\sin^4\theta\sin^2\phi\cos^2\phi\}\cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
&= \tfrac{2}{105}\pi c_{12}[(\boldsymbol{v}\cdot\boldsymbol{h})^2(15\boldsymbol{h}\otimes\boldsymbol{h} + 3\boldsymbol{i}\otimes\boldsymbol{i} + 3\boldsymbol{j}\otimes\boldsymbol{j})\\
&\quad + 6(\boldsymbol{v}\cdot\boldsymbol{h})(\boldsymbol{v}\cdot\boldsymbol{i})(\boldsymbol{h}\otimes\boldsymbol{i} + \boldsymbol{i}\otimes\boldsymbol{h}) + 6(\boldsymbol{v}\cdot\boldsymbol{h})(\boldsymbol{v}\cdot\boldsymbol{j})(\boldsymbol{h}\otimes\boldsymbol{j} + \boldsymbol{j}\otimes\boldsymbol{h})\\
&\quad + 2(\boldsymbol{v}\cdot\boldsymbol{i})(\boldsymbol{v}\cdot\boldsymbol{j})(\boldsymbol{i}\otimes\boldsymbol{j} + \boldsymbol{j}\otimes\boldsymbol{i}) + (\boldsymbol{v}\cdot\boldsymbol{i})^2(3\boldsymbol{h}\otimes\boldsymbol{h} + \boldsymbol{j}\otimes\boldsymbol{j} + 3\boldsymbol{i}\otimes\boldsymbol{i})\\
&\quad + (\boldsymbol{v}\cdot\boldsymbol{j})^2(3\boldsymbol{h}\otimes\boldsymbol{h} + 3\boldsymbol{j}\otimes\boldsymbol{j} + \boldsymbol{i}\otimes\boldsymbol{i}).
\end{aligned} \tag{A 6}$$

The integrands of (28), (30), and (32) may be expressed as functions of $\boldsymbol{k}$ by utilizing the approximate form of $f^{(2)}$ given by (38) and employing (5), (8) and the relation

$$C_1'^2 - C_1^2 = -(1+e)\,[(\boldsymbol{Q}-\boldsymbol{u})\cdot\boldsymbol{k}]\,(\boldsymbol{k}\cdot\boldsymbol{c}_{12}) - \tfrac{1}{4}(1-e^2)\,(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^2. \qquad \text{(A 7)}$$

where $\boldsymbol{Q} = \frac{1}{2}(\boldsymbol{c}_1+\boldsymbol{c}_2)$. Then, for example, (A 5) is used to establish that

$$\int (\boldsymbol{k}\cdot\boldsymbol{Dk})\,(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^3\,\mathrm{d}\boldsymbol{k} = \tfrac{1}{12}\pi c_{12}(3\boldsymbol{c}_{12}\cdot\boldsymbol{D}\boldsymbol{c}_{12} + c_{12}^2\,\mathrm{tr}\,\boldsymbol{D}); \qquad \text{(A 8)}$$

while (A 6) gives

$$\begin{aligned}\int (\boldsymbol{k}\cdot\boldsymbol{Dk})\,\boldsymbol{k}\otimes\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{c}_{12})^2\,\mathrm{d}\boldsymbol{k} = \tfrac{2}{105}\pi[&2(\mathrm{tr}\,\boldsymbol{D})\,\boldsymbol{c}_{12}\otimes\boldsymbol{c}_{12} + 2(\boldsymbol{c}_{12}\cdot\boldsymbol{D}\boldsymbol{c}_{12})\,\boldsymbol{I}\\ &+4(\boldsymbol{c}_{12}\otimes\boldsymbol{D}\boldsymbol{c}_{12} + \boldsymbol{D}\boldsymbol{c}_{12}\otimes\boldsymbol{c}_{12}) + (\mathrm{tr}\,\boldsymbol{D})\,\boldsymbol{I} + 2\boldsymbol{D}]. \qquad \text{(A 9)}\end{aligned}$$

After carrying out the integrations over $\boldsymbol{k}$, the integrations over $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, or alternatively over $\boldsymbol{Q}$ and $\boldsymbol{c}_{12}$, are performed in the fashion outlined by Chapman & Cowling (1970, §1.4). These are facilitated by noting that

$$f_{10}^{(1)}f_{20}^{(1)} = \frac{n^2}{(2\pi T)^3}\exp\left\{-\frac{1}{2T}[2(Q-u)^2 + \tfrac{1}{2}c_{12}^2]\right\}, \qquad \text{(A 10)}$$

$$\ln\frac{f_{20}^{(1)}}{f_{10}^{(1)}} = -\frac{1}{2T}(\boldsymbol{Q}-\boldsymbol{u})\cdot\boldsymbol{c}_{12}. \qquad \text{(A 11)}$$

Integrations over $\boldsymbol{k}$ that lead to integrals that vanish at this step have not been given above.

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